

DISPERSION RELATION FOR ALFVÉN WAVES IN A VISCOUS, DIFFUSIVE PLASMA

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Abstract. Propagation of Alfvén waves in the solar plasma has been a topic of scientific interest for a long time. We have derived a dispersion relation $\omega^4 + \omega^2[(\nu^2 + \eta^2)k^4 - v_A^2 k^2] + i\omega(\nu + \eta)v_A^2 k^4 + (\nu\eta v_A^2 k^6 + \nu^2 \eta^2 k^8) = 0$ for shear Alfvén waves in a viscous and diffusive plasma.

The MHD equations for viscous and diffusive plasma are

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \nabla) \vec{v} = \frac{1}{\mu}(\nabla \times \vec{B}) \times \vec{B} + \rho\nu \nabla^2 \vec{v} \quad \text{Momentum equation} \quad (1)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B} \quad \text{Induction equation} \quad (2)$$

$$\nabla \cdot \vec{B} = 0 \quad \text{Magnetic flux conservation} \quad (3)$$

where \vec{v} is the velocity, \vec{B} the magnetic field and ρ, μ, η, ν are, respectively, the mass density, magnetic permeability, magnetic diffusivity and the coefficient of viscosity. Let us consider small perturbations from the equilibrium [1]:

$$\rho = \rho_0 + \rho_1 \quad \vec{v} = \vec{v}_1 \quad \vec{B} = \vec{B}_0 + \vec{B}_1$$

and linearize the equations (1) through (3) by neglecting squares and products of the small quantities (denoted by subscript 1). Here, the quantities with subscript 0 represent their values in equilibrium. Notice that the equilibrium velocity v_0 is zero [1]. After linearization, we have the corresponding relations as the following:

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = \frac{1}{\mu}(\nabla \times \vec{B}_1) \times \vec{B}_0 + \rho_0 \nu \nabla^2 \vec{v}_1 \quad (4)$$

$$\frac{\partial \vec{B}_1}{\partial t} = \nabla \times (\vec{v}_1 \times \vec{B}_0) + \eta \nabla^2 \vec{B}_1 \quad (5)$$

$$\nabla \cdot \vec{B}_1 = 0 \quad (6)$$

Here, the magnetic field \vec{B}_0 is taken uniform as well as time independent. On differentiating equation (4) with respect to time and then substituting the expressions for $\partial \vec{v}_1 / \partial t$ and $\partial \vec{B}_1 / \partial t$ from equations (4) and (5), we get

$$\begin{aligned} \frac{\partial^2 \vec{v}_1}{\partial t^2} = v_A^2 \left\{ \nabla \times [\nabla \times (\vec{v}_1 \times \hat{B}_0)] \right\} \times \hat{B}_0 + \frac{\eta}{\mu \rho_0} \nabla \times (\nabla^2 \vec{B}_1 \times \vec{B}_0) \\ + \frac{\nu}{\mu \rho_0} \nabla^2 [(\nabla \times \vec{B}_1) \times \vec{B}_0] + \nu^2 \nabla^4 \vec{v}_1 \end{aligned} \quad (7)$$

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where \hat{B}_0 is the unit vector along \vec{B}_0 and $v_A (\equiv B_0/\sqrt{\mu\rho_0})$ is the Alfvén velocity. Let us seek a plane-wave solution of the form

$$\vec{v}_1 = \vec{v} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad \vec{B}_1 = \vec{B} e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (8)$$

where \vec{k} is the wave vector and ω the frequency. The effect of the plane-wave assumption is simply to replace $\partial/\partial t$ by $-i\omega$ and ∇ by $i\vec{k}$. Under the plane-wave assumption, equation (7) reduces to

$$\begin{aligned} \omega^2 \vec{v}_1 = v_A^2 \left\{ \vec{k} \times [\vec{k} \times (\vec{v}_1 \times \hat{B}_0)] \right\} \times \hat{B}_0 + \frac{i\eta k^2}{\mu\rho_0} (\vec{k} \cdot \vec{B}_0) \vec{B}_1 \\ + \frac{i\nu k^2}{\mu\rho_0} [(\vec{k} \times \vec{B}_1) \times \vec{B}_0] - \nu^2 k^4 \vec{v}_1 \end{aligned} \quad (9)$$

For plane-wave assumption, equation (6) gives

$$\vec{k} \cdot \vec{B}_1 = 0 \quad (10)$$

It shows that the magnetic field perturbation is normal to the propagation vector \vec{k} . Equation (9) can be simplified as

$$\begin{aligned} \omega^2 \vec{v}_1 = v_A^2 \left\{ (\vec{k} \cdot \hat{B}_0)(\vec{k} \cdot \hat{B}_0) \vec{v}_1 - (\vec{k} \cdot \hat{B}_0)(\vec{k} \cdot \vec{v}_1) \hat{B}_0 + \vec{k} [(\vec{k} \cdot \vec{v}_1) - (\hat{B}_0 \cdot \vec{v}_1)(\vec{k} \cdot \hat{B}_0)] \right\} \\ + \frac{i\eta k^2}{\mu\rho_0} (\vec{k} \cdot \vec{B}_0) \vec{B}_1 + \frac{i\nu k^2}{\mu\rho_0} [(\vec{B}_1 \cdot \vec{B}_0) \vec{k} - \vec{k} (\vec{B}_0 \cdot \vec{B}_1)] - \nu^2 k^4 \vec{v}_1 \end{aligned} \quad (11)$$

Let the propagation vector \vec{k} makes an angle θ with the equilibrium magnetic field \vec{B}_0 . Then equation (11) reduces to

$$\begin{aligned} \omega^2 \vec{v}_1 = v_A^2 \left\{ k^2 \cos^2 \theta \vec{v}_1 - k \cos \theta (\vec{k} \cdot \vec{v}_1) \hat{B}_0 + \vec{k} [(\vec{k} \cdot \vec{v}_1) - k \cos \theta (\hat{B}_0 \cdot \vec{v}_1)] \right\} \\ + \frac{i\eta k^3 B_0 \cos \theta}{\mu\rho_0} \vec{B}_1 + \frac{i\nu k^2}{\mu\rho_0} [k B_0 \cos \theta \vec{B}_1 - \vec{k} (\vec{B}_0 \cdot \vec{B}_1)] - \nu^2 k^4 \vec{v}_1 \end{aligned} \quad (12)$$

Multiplying equation (12) by \hat{B}_0 , we get

$$\omega^2 (\vec{v}_1 \cdot \hat{B}_0) = \frac{i\eta k^3 \cos \theta}{\mu\rho_0} (\vec{B}_1 \cdot \vec{B}_0) - \nu^2 k^4 (\vec{v}_1 \cdot \hat{B}_0)$$

The imaginary part disappears when we consider \vec{B}_1 orthogonal to \vec{B}_0 . Now, we get

$$(\omega^2 + \nu^2 k^4) (\hat{B}_0 \cdot \vec{v}_1) = 0 \quad (13)$$

Since $(\omega^2 + \nu^2 k^4)$ is not zero, we have

$$\hat{B}_0 \cdot \vec{v}_1 = 0 \quad (14)$$

It shows that the perturbation of velocity is normal to the ambient magnetic field. The discussion so far leads to the geometry as shown in Figure 1.

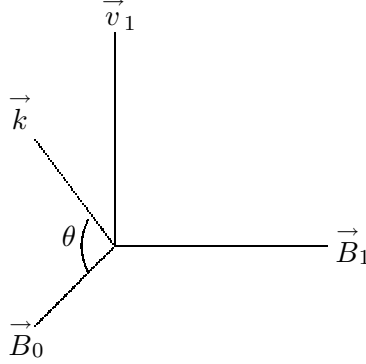


Figure 1: \vec{B}_0 , \vec{B}_1 and \vec{v}_1 are orthogonal to each other. \vec{k} lies in the (\vec{B}_0, \vec{v}_1) -plane and \vec{k} makes an angle θ with \vec{B}_0 .

Now, multiplying equation (12) by \vec{k} and using equations (10) and (14), we get

$$\omega^2(\vec{v}_1 \cdot \vec{k}) = v_A^2 k^2 (\vec{v}_1 \cdot \vec{k}) - \nu^2 k^4 (\vec{v}_1 \cdot \vec{k})$$

$$\left[\omega^2 - v_A^2 k^2 + \nu^2 k^4 \right] (\vec{k} \cdot \vec{v}_1) = 0 \quad (15)$$

Thus, we have either

$$\vec{k} \cdot \vec{v}_1 = 0 \quad (16)$$

or

$$\omega^2 - v_A^2 k^2 + \nu^2 k^4 = 0 \quad (17)$$

The relation (16) leads to the shear Alfvén waves whereas the relation (17) leads to the compressional Alfvén waves. For shear Alfvén waves, the perturbation is incompressible so that we have $\nabla \cdot \vec{v}_1 = 0$. Under the plane-wave assumption, this relation gives the equation (16).

Since $\vec{k} \cdot \vec{v}_1 = 0$, we have $\theta = 0$. Using equation (16) (with $\theta = 0$) in (12) (for Figure 1), we get

$$\omega^2 \vec{v}_1 = v_A^2 k^2 \vec{v}_1 + \frac{i(\nu + \eta)k^3 B_0}{\mu \rho_0} \vec{B}_1 - \nu^2 k^4 \vec{v}_1 \quad (18)$$

For the plane wave solution, equation (5) gives

$$(\eta k^2 - i\omega) \vec{B}_1 = iB_0 k \vec{v}_1 \quad (19)$$

From equations (18) and (19), we get a dispersion relation

$$\omega^4 + \omega^2[(\nu^2 + \eta^2)k^4 - v_A^2 k^2] + i\omega(\nu + \eta)v_A^2 k^4 + (\nu\eta v_A^2 k^6 + \nu^2 \eta^2 k^8) = 0 \quad (20)$$

This dispersion relation incorporates the viscosity as well as diffusivity.

For the plane wave solution, equation (4) gives

$$(\nu k^2 - i\omega) \vec{v}_1 = \frac{iB_0 k}{\mu \rho_0} \vec{B}_1 \quad (21)$$

From equations (18) and (21), we get a dispersion relation

$$\omega^2 = k^2 \left[v_A^2 - i\omega(\nu + \eta) \right] + \nu\eta k^4$$

This dispersion relation is the same as obtained by Pekünlü et al. [2] and can also be obtained by using the equations (21) and (19).

We have thus obtained a new dispersion relation (20).

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